Name: Key

- This packet is due at **12:00 pm PDT (noon) on Friday, April 9**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- When your final answer to a problem is a sentence, paragraph, expression, equation, or number, you must **draw a box/circle/cloud/[your favorite enclosing shape here] around it**.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- **Please Note:** You may need to download a copy of the textbook in order to view the page numbers.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Carefully read pages 6-9, starting at "Sigma (Summation) Notation" and do the following:

   (a) State whether the given sums are equal or unequal. You do not need to show your work.

   i. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$.

   $1+2+\cdots+10$ $\quad$ $1+2+\cdots+10$

   Equal

   ii. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$ $=$ $1\cdot 0 + 2\cdot 1 + \cdots + 10\cdot 9$

   $1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + 10\cdot$

   Equal

   (b) Suppose $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. Compute the following sums. You do not need to show your work.

   i. $\sum_{i=1}^{100} (a_i + b_i)$ $= \sum_{i=1}^{100} a_i + \sum_{i=1}^{100} b_i = 15 - 12 = \boxed{3}$

   ii. $\sum_{i=1}^{100} (3a_i - 4b_i)$ $= \sum_{i=1}^{100} 3a_i - \sum_{i=1}^{100} 4b_i$

   $= 3\sum_{i=1}^{100} a_i - 4\sum_{i=1}^{100} b_i$

   $= 3(15) - 4(-12) = \boxed{93}$

(c) Use summation properties and formulas to rewrite and evaluate the following sums:

i. $\sum_{i=5}^{10} i$ $= \sum_{i=1}^{10} i - \sum_{i=1}^{4} i$

$= \frac{10(10+1)}{2} - \frac{4(4+1)}{2} = \boxed{45}$

ii. $\sum_{j=1}^{50}(j^2 - 2j)$ $= \sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j$

$= \frac{50(50+1)(100+1)}{6} - 2 \cdot \frac{50(50+1)}{2}$

$= 42925 - 2550 = \boxed{40375}$

iii. $\sum_{j=11}^{20}(j^2 - 10j)$ $= \sum_{j=11}^{20} j^2 - 10\sum_{j=11}^{20} j$

$= \sum_{j=1}^{20} j^2 - \sum_{j=1}^{10} j^2 - 10\left(\sum_{j=1}^{20} j - \sum_{j=1}^{10} j\right)$

$= \frac{20(20+1)(40+1)}{6} - \frac{10(10+1)(20+1)}{6} - 10\left(\frac{20\cdot 21}{2} - \frac{10\cdot 11}{2}\right)$

$= \boxed{935}$

iv. $\sum_{k=1}^{25}[(2k)^2 - 100k]$

$\sum_{k=1}^{25} 4k^2 - 100k = 4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k$

$= 4 \cdot \frac{25\cdot 26\cdot 51}{6} - 100 \cdot \frac{25\cdot 26}{2}$

$= \boxed{-10400}$

2. **Important review of absolute value:** Recall the *absolute value function* is defined for all real x by

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

A good way of thinking about the absolute value is that $|x|$ means "the size of x." Note that $x \leqslant |x|$ is true for every x. Below are some additional facts you should never forget. Suppose $r \geqslant 0$.

- The equation "$|x| = r$" means "$x = r$ or $x = -r$", which in set notation means $x \in \{-r, r\}$.
- The inequality "$|x| < r$" means "$x < r$ and $-x < r$". Since "$-x < r$" is equivalent to "$x > -r$", then "$x < r$ and $-x < r$" can be conveniently written together as "$-r < x < r$". In set notation this means $x \in (-r, r)$.
- Similarly, "$|x| \leqslant r$" means "$x \leqslant r$ and $-x \leqslant r$", or more simply, "$-r \leqslant x \leqslant r$". In set notation this means $x \in [-r, r]$.
- The inequality "$|x| > r$" means "$x > r$ or $-x > r$", or equivalently "$x > r$ or $x < -r$". Notice that the "or" here is exclusive because the number x cannot be greater than r and less than $-r$ simultaneously (because $r \geqslant 0$). In set notation, "$|x| > r$" means $x \in (-\infty, -r) \cup (r, \infty)$.
- Similarly, "$|x| \geqslant r$" means "$x \geqslant r$ or $-x \geqslant r$", or equivalently "$x \geqslant r$ or $x \leqslant -r$". This is also an exclusive "or" unless $x = r = 0$. In set notation, "$|x| \geqslant r$" means $x \in (-\infty, -r] \cup [r, \infty)$.

We can summarize the facts above as follows. Suppose $r \geqslant 0$. Then:

$$\begin{array}{lcll} |x| = r & \iff & x = r \text{ or } x = -r & \iff & x \in \{-r, r\}\ , \\ |x| < r & \iff & -r < x < r & \iff & x \in (-r, r)\ , \\ |x| \leqslant r & \iff & -r \leqslant x \leqslant r & \iff & x \in [-r, r]\ , \\ |x| > r & \iff & x < -r \text{ or } x > r & \iff & x \in (-\infty, -r) \cup (r, \infty)\ , \\ |x| \geqslant r & \iff & x \leqslant -r \text{ or } x \geqslant r & \iff & x \in (-\infty, -r] \cup [r, \infty)\ . \end{array}$$

Write the largest interval (or union of intervals) of x values satisfying the given inequality.

(a) $|x| < 7$ $x \in (-7, 7)$

(b) $|x| > 7$ $x \in (-\infty, -7) \cup (7, \infty)$

(c) $|x| > -1$ (This inequality is true for *many* values of x...) $x \in (-\infty, \infty)$

(d) $|x - 5| < 13$ $\hookleftarrow$ $x - 5 \in (-13, 13)$ $\hookleftarrow$ $x \in (-8, 18)$

(e) $|2x + 6| \geqslant 9$ $\hookleftarrow$ $2x + 6 \in (-\infty, -9] \cup [9, \infty)$ $\hookleftarrow$ $2x \in (-\infty, -15] \cup [3, \infty)$ $\hookleftarrow$ $x \in (-\infty, -\frac{15}{2}] \cup [\frac{3}{2}, \infty)$

(f) $|\frac{x-1}{3}| < 4$ $\hookleftarrow$ $\frac{x-1}{3} \in (-4, 4)$ $\hookleftarrow$ $x - 1 \in (-12, 12)$ $\hookleftarrow$ $x \in (-11, 13)$

3. An important feature of the absolute value function is that it provides us with a notion of *distance* between real numbers. That is, if x and y are two real numbers, then

$$|x-y| = \text{the distance between } x \text{ and } y.$$

In this course we will see inequalities of the form $|x-c|<\varepsilon$ (where $\varepsilon>0$) *very often*, so it is vital that you understand the following equivalence of statements:

$$\begin{aligned} x \text{ is less than } \varepsilon \text{ away from } c &\iff |x-c|<\varepsilon \\ &\iff -\varepsilon < x-c < \varepsilon \\ &\iff c-\varepsilon < x < c+\varepsilon \\ &\iff x \in (c-\varepsilon, c+\varepsilon). \end{aligned}$$

Use the above equivalences to find four alternative ways to write each of the following statements:

(a) x is less than 0.001 away from 6

$|x-6| < 0.001$

$-0.001 < x-6 < 0.001$

$5.999 < x < 6.001$

$x \in (5.999, 6.001)$

(b) $-14 < x < 18$

$|x-2| < 16$

$-16 < x-2 < 16$

x is less than 16 away from 2

$x \in (-14, 18)$

(c) $|x-\pi| < 2\pi$

x is less than 2π away from π

$-2\pi < x-\pi < 2\pi$

$-\pi < x < 3\pi$

$x \in (-\pi, 3\pi)$

(d) $x \in (-59, 1)$

$|x-(-29)| < 30$

$-30 < x-(-29) < 30$

x is less than 30 away from -29

$-59 < x < 1$

4. **The Triangle Inequality:** Another extremely important feature of the absolute value function is the following inequality, which is known as the *Triangle Inequality*:

$$|x+y| \leqslant |x| + |y| \qquad \text{for all real } x \text{ and } y.$$

Use the triangle inequality to explain why the following is true:

$$\textit{If } |a-100| < 7 \textit{ and } |b-100| < 9, \textit{ then } |a-b| < 16.$$

Hint: Note that $a - b = (a - 100) + (100 - b)$.

$$|a-b| = |(a-100) + (100-b)| \leq |a-100| + |100-b|$$
$$< 7 + 9 = 16$$

5. **Factorials:** Suppose n is a positive integer (i.e., $n \in \{1, 2, 3, 4, \dots\}$). As we will soon see, we will often need to multiply the numbers $1, 2, 3, \dots, n$ together. We call the resulting product n *factorial*, which is written as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

It will be convenient to extend this definition to the $n = 0$ case as well, by defining 0! to be 1.

Suppose $n > 25$. Rearrange the following quantities so that they increase from left to right.

$$10^n, \qquad 10^{-n}, \qquad e^n, \qquad e^{-3n}, \qquad n^n, \qquad \ln(n), \qquad \log_{10}(n), \qquad n!, \qquad n$$

(*Hint*: It would be a really good idea to use Desmos to graph the curves $y = 10^n$, $y = 10^{-n}$, $y = e^n$, $\dots$ on the same screen in order to make some of the comparisons easier. It will do something that may seem strange with $y = n!$ for negative values of n, but there is no cause for concern.)

$$e^{-3n} < 10^{-n} < \log_{10}(n) < \ln(n) < n < e^n < 10^n < n! < n^n$$

6. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues:

   (a) $\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots\}$

   $a_n = \left(\frac{1}{3}\right)^n$ for $n \geq 0$

   (b) $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\}$

   $a_n = \frac{(-1)^n n}{(n+1)^2}$ for $n \geq 1$

7. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = \frac{n^3}{n^3+1} = \frac{n^3}{n^3\left(1+\frac{1}{n^3}\right)} = \frac{1}{1+\frac{1}{n^3}}$ converges

   $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = 1$

   (b) $a_n = \frac{3^{n+2}}{5^n} = \frac{3^n \cdot 3^2}{5^n} = 9 \cdot \left(\frac{3}{5}\right)^n$ converges

   $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 9 \cdot \left(\frac{3}{5}\right)^n = 0$

   (c) $\{n^2 e^{-n}\}$ converges

   $\lim_{n\to\infty} n^2 e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n} = \lim_{n\to\infty} \frac{2n}{e^n} = \lim_{n\to\infty} \frac{2}{e^n} = 0$ (L'Hopital)

   (d) $\{n\cos(n\pi)\} = \{-1, 2, -3, 4, -5, 6, \dots\}$

   diverges

8. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

   (a) $a_n = \frac{1}{2n+3}$ $\left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\right\}$

   monotonic decreasing after $n_0 = 0$

   bounded above by $\frac{1}{3}$ and below by 0, so a_n is bounded

   (b) $a_n = n(-1)^n$

   $\{-1, 2, -3, 4, -5, \dots\}$

   not monotonic, not increasing, not decreasing, neither bounded above nor below

9. Consider the sequence defined by $a_n = 7 \cdot (\frac{1}{5})^n + 5$.

   (a) Find the limit L of this sequence.

$\lim_{n\to\infty} 7\left(\frac{1}{5}\right)^n + 5 = 5$

   (b) Find an integer n_0 that has the property that $|a_n - L| < 0.01$ for every $n \geqslant n_0$.

Want: $|a_{n_0} - 5| < 0.01$

LHS: $|a_{n_0} - 5| = |7(\frac{1}{5})^{n_0} + 5 - 5| = |7(\frac{1}{5})^{n_0}| = 7 \cdot (\frac{1}{5})^{n_0}$

Want: $7(\frac{1}{5})^{n_0} < 0.01 \leftrightarrow \frac{7}{5^{n_0}} < 0.01 \leftrightarrow \frac{7}{0.01} < 5^{n_0} \leftrightarrow \log_5(\frac{7}{0.01}) < n_0$

$4.07 < n_0$

$\downarrow$

$n_0 = 5$

   (c) Find an integer n_0 that has the property that $|a_n - L| < 0.0001$ for every $n \geqslant n_0$.

Want: $7(\frac{1}{5})^{n_0} < 0.0001 \leftrightarrow \frac{7}{0.0001} < 5^{n_0}$

$\leftrightarrow \log_5(\frac{7}{0.001}) < n_0$

$\leftrightarrow 6.93 < n_0 \rightarrow n_0 = 7$

10. Suppose $\{a_n\}$ is a convergent sequence with $\lim_{n\to\infty} a_n = 10$. Let $\{b_n\}$ be the sequence whose terms are given by

$$\{a_1, a_1, a_2, a_2, a_3, a_3, \dots\}$$

   (a) Find the value of $L = \lim_{n\to\infty} b_n$.

$\lim_{n\to\infty} b_n = 10$

   (b) Suppose $|a_n - 10| < 0.05$ for every $n \geqslant 1500$. Find an integer n_0 that has the property that $|b_n - L| < 0.05$ for every $n \geqslant n_0$.

$n_0 = 3000$ works here because $b_{3000} = a_{1500}$,
Since every a_k after a_{1500} is within 0.05 of 10,
then every b_n after b_{3000} is within 0.05 of 10.

11. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence. For each $n \geqslant 1$, define

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

and note that the sums $S_1, S_2, S_3, \ldots$ form another sequence $\{S_n\}_{n=1}^{\infty}$. A substantial part of this course amounts to studying the relationships between $\{a_k\}_{k=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$. To begin understanding these relationships, let $S_0 = 0$ and note that we can shift the index of the original sequence $\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, S_3, \ldots\}$ backward by 1 to obtain a nearly identical sequence: $\{S_{n-1}\}_{n=1}^{\infty} = \{S_0, S_1, S_2, \ldots\}$.

(a) Suppose there is a number L such that $\lim_{n\to\infty} S_n = L$. What can you say about $\lim_{n\to\infty} S_{n-1}$?

S_{n-1} is the same sequence as S_n, so

$$\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = L$$

(b) For each $n \geqslant 1$, what is $S_n - S_{n-1}$?

$$S_n - S_{n-1} = (a_1 + \cdots + a_n) - (a_1 + \cdots + a_{n-1})$$

$$= a_n$$

(c) Suppose $\{S_n\}_{n=1}^{\infty}$ converges. What must be true about $\lim_{k\to\infty} a_k$?
Hint: Think about $\lim_{n\to\infty} S_n - S_{n-1}$ in two different ways

$$\lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0$$

$$\lim_{n\to\infty} a_n = 0$$